Unitary moduli schemes: smooth case

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1 Introduction

We will start by discussing a 'baby version' of section 4 of [2], and hope that this will provide some intuition for what in the paper. Let $E/\mathbb{Q}$ be a quadratic imaginary field, let V/E be an hermitian space of rank 5 with signature $(1,4)$ and let $GU(V)$ be the associated unitary similitude group. Furthermore let $p>2$ be a prime inert in E and let $K^p \subset GU(V)$ be a neat compact open subgroup. We will consider the functor $\mathbf{M} = \mathbf{M}(V, K^p)$ which associates to $S/\mathbb{Z}_{(p)}$ equivalence classes of triples (A, λ, η^p), where

- (A,λ) is a unitary $\mathcal{O}_E$ abelian scheme over S of signature type $(1,4)$ and λ is a prime-to-p polarisation;
- η^p is a K^p level structure (which is something that we will be vague about throughout this talk)

Proposition 1.0.1. *The functor $\mathbf{M}$ is representable by a scheme $\mathbf{M}$ and the structure map $\mathbf{M} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is quasiprojective and smooth of relative dimension 4.*

Vollaard and Wedhorn [3] studied the basic locus of this Shimura variety (with 5 replaced by arbitrary n) and proved the following result:

Theorem 1.0.2 (Theorem 5.2 of [3])**.** *Assume that K^p is sufficiently small, then the irreducible components of the basic locus $\mathbf{M}^b_{\overline{\mathbb{F}}_p}$ are isomorphic to a certain Deligne-Lusztig variety $DL(W)$, which is an irreducible smooth projective variety.*

- *Moreover, the set of irreducible components of the basic locus is in bijection with the adelic double coset*

$$I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K,$$

  where $I/\mathbb{Q}$ is the unique inner form of G such that $G(\mathbb{A}^\infty) \simeq I(\mathbb{A}^\infty)$ and such that $I(\mathbb{R})$ is compact mod centre.

- *Two irreducible components X_1 and X_2 are either disjoint, intersect in a point or intersect in a smooth curve $C \subset DL(W)$. This intersection behaviour is governed by the Bruhat-Tits tree of*

$I_{\mathbb{Q}_p} = J_b$, *in particular the correspondences*

$$T_2 : X \mapsto \sum_{Y | X\cap Y = \{pt\}} Y$$
$$T_1 : X \mapsto \sum_{Y | X\cap Y = C} Y$$

on $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K$ *can be identified with 'natural' Hecke operators* T_1, T_2 *in the spherical hecke algebra of* I.

Remark 1.0.3. This means that the natural map

$$\beta : \mathbb{Q}_\ell[I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K] \simeq H_4^{\mathrm{BM}}(\mathbf{M}^{\mathrm{b}}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \to H^4_c(\mathbf{M}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \to H^4_c(\mathbf{M}^{\mathrm{b}}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell[I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K]$$

is a linear combination

$$\sum_{\delta=0}^{2} \mathbf{d}_{2-\delta} \cdot T_i$$

with the coefficients given by intersection numbers (it turns out the the intersection numbers between X and Y only depend on the codimension of $X \cap Y$ and not on the actual choice of components X and Y, this uses the excess intersection formula and some deformation theory). Here T_0 is the identity and its coefficient will be the self-intersection of components X in $\mathbf{M}_{\overline{\mathbb{F}}_p}$.

Remark 1.0.4. The main theorem of [4] also tells us what β is, but now on the other side of the Satake correspondence. To be precise, they interpret the spherical Hecke algebra as global functions on the stack

$$\left[\frac{\hat{G}\sigma}{\mathrm{Ad}\,\hat{G}}\right]$$

of unramified Langlands parameters and write α (up to a scalar) as a product of

$$\prod_{\chi \in \Phi^\vee_{\mathrm{rel}}} (e^\chi \pm 1)$$

of certain cocharacters of $\hat{S}$, where S is a maximal split torus of G. This is what allows them to say for what kind of (spherical) representations π_p of $G(\mathbb{Q}_p)$ the map β is nonzero, in terms of the Langlands parameter of π_p. In particular, they prove that it suffices to show that the Langlands parameter,which is just a conjugacy class of elements in $\hat{G}(\overline{\mathbb{Q}_\ell})$, is regular semi-simple. It is possible to translate from this point of view to the previous by doing lots of combinatorics (appendix B of [2]) and we then find [I probably made some mistakes doing this computation]

$$\mathbf{d}_0 = p^6 + 2p^5 - p^4 + p^3 - p^2 - p + 1$$
$$\mathbf{d}_1 = p^3 - 4p^2 + p$$
$$\mathbf{d}_2 = 1.$$

Remark 1.0.5. The proof in [3] proceeds by studying the Rapoport-Zink space that uniformises $\mathbf{M}^{\mathrm{b}}_{\overline{\mathbb{F}}_p}$, which has the disadvantage that one does not use the description of $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K$ as a moduli space. The authors of [2] proceed by giving a moduli description of $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K$ and then writing down a moduli theoretic correspondence

$$\begin{array}{ccc} & \mathbf{B}_{\mathfrak{p}} & \\ \swarrow & & \searrow \\ I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/K & & \mathbf{M}^{\mathrm{b}}_{\overline{\mathbb{F}}_p}. \end{array}$$

THey then identify the right arrow with the normalisation map and the fibers of the left arrow with Deligne-Lusztig varieties. This is also the perspective taken in [1], which treats a slightly different case. This makes it easier to compute normal bundles/tangent bundles which is necessary to use the excess intersection formula.

2 Unitary Moduli schemes

In this section, we will follow Section 4 of [2] and construct the analogues of the objects from the introduction. We start by fixing

2.1 Notation

- a CM extension F/F^+;
- a CM type Φ containing τ_∞;
- a special inert prime $\mathfrak{p}$ of F^+;
- A rational skew-hermitian space W_0 over $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ with similitude group T_0;
- a neat open compact subgroup $K_0^p \subset T_0(\mathbb{A}^{\infty,p})$;
- A totally positive element $\varpi \in \mathcal{O}_{F^+}$ with $\mathfrak{p}$-adic valuation one and $\mathfrak{q}$-adic valuation zero for all $\mathfrak{p} \neq \mathfrak{q} \mid p$;
- A standard indefinite hermitian space V over F of rank $N \geq 1$
- For every prime $\mathfrak{q} \mid p$, a self dual lattice $\Lambda_{\mathfrak{q}}$ in $V \otimes_F F_{\mathfrak{q}}$ [This is basically the same thing as a choice of hyperspecial subgroup of $U(V)(\mathbb{Q}_p)$, and these don't always exist when N is even]

2.2 A unitary moduli scheme

Recall that we have a finite étale scheme $\mathbf{T}_{\mathfrak{p}}/\mathbb{Z}_p^{\Phi}$ with an action of the abelian group

$$T_0(\mathbb{A}^{\infty,p})/T_0(\mathbb{Z}_{(p)})K_0^p.$$

The S-points $\mathbf{T}_{\mathfrak{p}}(S)$ have a moduli description as equivalence classes of triples $(A_0, \lambda_0, \eta_0^p)$ where A_0 is a unitary $\mathcal{O}_F$-abelian scheme of signature type Φ, where λ_0 is a prime-to-p polarisation and where η_0^p is a K_0^p level structure. All the moduli schemes that we define in this section will live over $\mathbf{T}_{\mathfrak{p}}$.

Definition 2.2.1. *Consider the moduli functor* $\mathbf{M}_{\mathfrak{p}} = \mathbf{M}_{\mathfrak{p}}(V, K^p)$*, for a fixed neat open compact subgroup* $K^P \subset U(V)(\mathbb{A}^{\infty,p})$*, such that* $\mathbf{M}_{\mathfrak{p}}(S)$ *is the set of equivalence classes of sextuples*

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p)$$

where

- $(A_0, \lambda_0, \eta_0^p) \in \mathbf{T}_{\mathfrak{p}}(S)$
- (A, λ) *is a unitary* $\mathcal{O}_F$ *abelian scheme of signature type* $N\Phi - \tau_\infty + \tau_\infty^c$ *over* S *with* λ *a prime-to-p polarisation*
- η^p *is a* K^p*-level structure, which is something like a* K^p *orbit of morphisms*

$$V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to \mathrm{Hom}^{\lambda_0,\lambda}_{F\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}} \left(H_1^{\acute{e}t}(A_0, \mathbb{A}^{\infty,p}), H_1^{\acute{e}t}(A, \mathbb{A}^{\infty,p}) \right).$$

Remark 2.2.2. In [2], the authors consider the objects $\mathbf{M}_{\mathfrak{p}}(V, -)$ as functors from

$$\mathfrak{k}(V)^p \times \mathfrak{T} \to \mathrm{Sch}'_{/\mathbb{Z}_p^\Phi},$$

essentially it says that all our constructions are compatible with changing the prime-to-p level (and that there are Hecke operators) and acting by the abelian group $T_0(\mathbb{A}^{\infty,p})/T_0(\mathbb{Z}_{(p)})K_0^p$, we refer to loc. cit. for the details

Remark 2.2.3. We are purposely omitting details regarding the equivalence relation on the sextuples and also the detailed definition of prime-to-p level structures, we refer to [2] for the details.

Theorem 2.2.4 (Thm 4.13 of [2])**.** *The morphism* $\mathbf{M}_{\mathfrak{p}} \to \mathbf{T}_{\mathfrak{p}}$ *is representable by quasi-projective smooth schemes of relative dimension* $N-1$*, the morphism is projective if and only its generic fiber is. The relative tangent sheaf is given by*

$$\mathcal{H}om\left(\omega_{\mathcal{A}^t,\tau_\infty}, (\mathrm{Lie}\, A)_{\tau_\infty}\right).$$

Moreover, there is an isomorphism

$$\mathbf{M}_{\mathfrak{p}}^\eta \simeq \mathrm{Sh}(V) \times_{\mathrm{Spec}\, F} \mathbf{T}_{\mathfrak{p}}^\eta$$

Remark 2.2.5. The Shimura variety $\mathrm{Sh}(V)$ is not of PEL type, so we cannot expect a moduli description without the 'extra torus part' provided by $\mathbf{T}_{\mathfrak{p}}$.

2.3 Basic correspondence

In this section, we will define a moduli interpretation $\mathbf{S}_{\mathfrak{p}}$ of the Shimura set that parametrises the irreducible components of the basic locus of $\mathbf{M}_{\mathfrak{p}}$. Using this, we will define a moduli theoretic correspondence

$$\begin{array}{ccc} & \mathbf{B}_{\mathfrak{p}} & \\ \swarrow & & \searrow \\ \mathbf{M}_{\mathfrak{p}}^{\mathrm{basic}} & & \mathbf{S}_{\mathfrak{p}}, \end{array} \tag{1}$$

and show that the left hand morphism can be identified with the normalisation of $\mathbf{M}_{\mathfrak{p}}^{\text{basic}}$ and the right hand morphism is surjective with geometric fibers isomorphic to certain Deligne-Lusztig varieties.

Definition 2.3.1. *We define $\mathbf{S}_{\mathfrak{p}} = \mathbf{S}_{\mathfrak{p}}(V, K^p)$ to be the moduli functor whose S points $\mathbf{S}_{\mathfrak{p}}(S)$ is the set of equivalence classes of sextuples*

$$(A_0, \lambda_0, \eta_0^p; A^\star, \lambda^\star, \eta^{p\star})$$

where

- $(A_0, \lambda_0, \eta_0^p) \in \mathbf{T}_{\mathfrak{p}}(S)$*;*
- $(A^\star, \lambda^\star)$ *is a unitary $\mathcal{O}_F$ abelian scheme of signature type $N\Phi$ over S such that* $\ker \lambda^\star[p^\infty]$ *is trivial if N is odd and contained in $A^\star[\mathfrak{p}]$ of rank p^2 if N is even;*
- $\eta^{p\star}$ *is a K^p level structure, which is something like a K^p orbit of morphisms*

$$\eta^{p\star} : V \otimes_{\mathbb{Q}}^{\infty,p} \to \mathrm{Hom}_{F\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}}^{\varpi\lambda_0,\lambda^\star} \left(H_1^{\acute{e}t}(A_0, \mathbb{A}^{\infty,p}), H_1^{\acute{e}t}(A, \mathbb{A}^{\infty,p}) \right).$$

Remark 2.3.2. The level structure at $\mathfrak{p}$ of this 'Shimura variety' really corresponds to a lattice $\Lambda \subset V \otimes_F F_{\mathfrak{p}}$ such that

$$p\Lambda \subset \Lambda^\vee$$

of index 0 (if N is odd) or index p^2 if N is even [this is what comes out of the Vollaard-Wedhorn construction], hence why we take level structures with respect to $\varpi\lambda_0$.

Proposition 2.3.3. *The morphism $\mathbf{S}_{\mathfrak{p}} \to \mathbf{T}_{\mathfrak{p}}$ is represented by finite étale schemes.*

Definition 2.3.4. *We define $\mathbf{B}_{\mathfrak{p}} = \mathbf{B}_{\mathfrak{p}}(V, K^p)$ to be the moduli functor whose S points $\mathbf{S}_{\mathfrak{p}}(S)$ is the set of equivalence classes of tuples*

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p; A^\star, \lambda^\star, \eta^{p\star}; \alpha),$$

where

- $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \in \mathbf{M}_{\mathfrak{p}}(S)$
- $(A_0, \lambda_0, \eta_0^p; A^\star, \lambda^\star, \eta^{p\star}) \in \mathbf{S}_{\mathfrak{p}}(S)$*.*
- $\alpha : A \to A^\star$ *is an $\mathcal{O}_F$-linear quasi-isogeny such that*
  - $\ker \alpha[p^\infty] \subset A[\mathfrak{p}]$
  - $\varpi\lambda = \alpha^\vee \circ \lambda^\star \circ \alpha$
  - α *preserves K^p level structures*

Remark 2.3.5. There is an obvious correspondence as in (1) by the first two conditions. Note that the condition $\varpi\lambda = \alpha^\vee \circ \lambda^\star \circ \alpha$ implies that $\ker \alpha[p^\infty] \subset A[\mathfrak{p}]$ is an isotropic subspace for the Weil pairing that satisfies

$$\mathrm{rk} \left(\ker \alpha[p^\infty])^\perp / (\ker \alpha[p^\infty]) \right) \simeq \begin{cases} 0 & \text{if } N \text{ odd} \\ p^2 & \text{if } N \text{ even} \end{cases}$$

2.3.6 Linear algebraic description of the fibers

The idea is now that the fiber $\mathbf{B}_{s^\star}$ of $\mathbf{B}_\mathfrak{p} \to \mathbf{S}_\mathfrak{p}$ over a point $s^* = (A^\star, \lambda^\star) \in \mathbf{S}_\mathfrak{p}(k)$ should have a linear algebraic description. For this we consider

$$\mathcal{V}_{s^\star} := H_1^{\mathrm{dR}}(A^*/k)_{\tau_\infty}$$

which we equip with the pairing

$$\{\ ,\ \}_{s^\star} : \mathcal{V}_{s^\star} \times \mathcal{V}_{s^\star} \to k$$

defined by $\{\ ,\ \}_{s^\star}(x, y) = \langle Fx, y\rangle_{\lambda^\star}$ where

$$F : H_1^{\mathrm{dR}}(A^*/k)_{\tau_\infty} \to H_1^{\mathrm{dR}}(A^*/k)_{\tau_\infty^c}$$

and where $\langle\ ,\ \rangle_{\lambda^\star}$ is the pairing

$$H_1^{\mathrm{dR}}(A^*/k)_{\tau_\infty^c} \times H_1^{\mathrm{dR}}(A^*/k)_{\tau_\infty} \to k$$

induced by $\lambda^\star$. Given (A, λ) and an isogeny

$$\alpha : A \to A^\star$$

with dual isogeny $\beta : A^\star \to A$ satisfying $\beta \circ \alpha = \varpi$, we consider the subspace

$$H = \beta_{*,\tau_\infty}^{-1}\left(\omega_{A^\vee,\tau_\infty}\right) \subset H_1^{\mathrm{dR}}(A^\star)_{\tau_\infty},$$

where

$$\beta_{*,\tau} : H_1^{\mathrm{dR}}(A^\star)_{\tau_\infty} \to H_1^{\mathrm{dR}}(A)_{\tau_\infty}$$

is the induced map on de-Rham homology. The main idea of everything that will happen next, is that the subspace H will uniquely determine β and therefore α. Therefore, the fiber $\mathbf{B}_{s^\star}$ will be contained in some sort of flag variety, and so we need to determine its image.

Lemma 2.3.7. *The subspace $H \subset \mathcal{V}_{s^\star}$ satisfies*

$$H^\perp \subset H,$$

where $H^\perp$ is the right-orthogonal complement of H under $\{\ ,\ \}_{s^\star}$. Moreover H has rank $\lceil \frac{N+1}{2} \rceil$.

Proof. Omitted, the proof in [2] is derived from Lemma 3.4.13 of op.cit. which mostly uses Dieudonné theory. ☐

Proposition 2.3.8 (Proposition A.1.3 of [2])**.** *There is a smooth projective geometrically connected scheme $DL_{s^\star}/k$ of dimension $\lfloor \frac{N-1}{2} \rfloor$ such that $DL_{s^\star}(S)$ parametrises rank $\lceil \frac{N+1}{2} \rceil$ sub-bundles $H \subset \mathcal{V}_{s^\star} \otimes_k \mathcal{O}_S$ such that $H^\perp \subset H$. Its tangent bundle is given by*

$$\mathrm{H}om(\mathcal{H}/\mathcal{H}^\perp, \mathcal{V}_{DL_{s^\star}}/\mathcal{H}).$$

Proof. The argument for representability is standard and shows that our variety is a closed subscheme of some Grassmannian, hence projective. If $R \twoheadrightarrow R_0$ is a surjection of artinian local rings with kernel I satisfying $I^2 = 0$ and $H_0 \in DL_{s^*}(R_0)$ then we can a lift $H \in DL_{s^*}(R)$ as follows:

- We first lift H_0 to an arbitrary subspace $H' \subset \mathcal{V}_R$ using smoothness of Grassmannians.
- We then note that since $I^p = 0$, the orthogonal complement $K = H'^{\perp}$ does not depend on the choice of lift of H' (because $H'^{\perp}$ depends only on $H'^{(p)}$ which does not depend on the choice of H');
- We are now free to choose a lift H of H_0 such that $K \subset H$, by the smoothness of $\mathbb{P}(\mathcal{V}/H^{\perp})$.

As a corollary we get the description of the tangent bundle from the theorem ☐

Theorem 2.3.9 (Theorem 4.2.5 of [2])**.** *The fibers $B_{s^\star}$ of $\mathbf{B}_{\mathfrak{p}} \to \mathbf{S}_{\mathfrak{p}}$ are isomorphic to $DL_{s^\star}$ with morphism described on k points as above. The morphism $B_{s^\star} \to \mathbf{B}_{\mathfrak{p}} \to \mathbf{M}_{\mathfrak{p}}$ is a closed immersion of pure codimension $\lfloor \frac{N}{2} \rfloor$ when K^p is sufficiently small with normal bundle*

$$\mathrm{H}om(\omega_{A^\vee,\tau_\infty}, \mathrm{im}_{\alpha_*,\tau_\infty}).$$

Remark 2.3.10. It should follow as in [3] that the the map $\mathbf{B}_{\mathfrak{p}} \to \mathbf{M}_{\mathfrak{p}}$ surjects onto the basic locus, but as far as I can tell this is never made explicit in [2]. [The fact that the map lands in the basic locus follows from the signature condition on $\mathbf{S}_{\mathfrak{p}}$.]

2.4 Uniformisation data

We are now going to relate $\mathbf{S}_{\mathfrak{p}}$, to a Shimura set for a definite inner form I of $U(V)$.

Definition 2.4.1. *A definite uniformisation datum for V at $\mathfrak{p}$ is a collection $V^\star, i, \{\lambda_{\mathfrak{q}^\star}\}_{\mathfrak{q}|p})$ where*

- *$V^\star$ is a standard definite hermitian space over F of rank N;*
- *$i : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ is an isometry;*
- *for every prime $\mathfrak{p} \neq \mathfrak{q}$ of F^+ above p, a self dual lattice $\Lambda^\star_{\mathfrak{q}}$;*
- *A lattice $\Lambda^\star_{\mathfrak{p}}$ such that*

  $$p\Lambda^\star_{\mathfrak{p}} \subset (\Lambda^\star_{\mathfrak{p}})^\vee$$

  of index 0 when N is odd and of index p^2 when N is even.

Remark 2.4.2. Such definite uniformisation data should always exist (by some kind of Hasse principle?)

Proposition 2.4.3. *There is an isomorphism*

$$\mathbf{S}_{\mathfrak{p}}(\overline{\mathbb{F}}_p) \simeq \mathrm{Sh}(V^\star, K^\star_p) \times \mathbf{T}_{\mathfrak{p}}(\overline{\mathbb{F}}_p).$$

2.5 Tate cycles

Suppose that $n = 2r + 1$ is odd, then we consider the natural maps

$$\mathrm{inc}_!^\star : \mathbb{Q}_\ell[\mathrm{Sh}(V^\star, K_p^\star)] \simeq H^0_{\mathfrak{T}}(\mathbf{S}_\mathfrak{p}, \mathbb{Q}_\ell) \simeq H^0_{\mathfrak{T}}(\mathbf{B}_\mathfrak{p}, \mathbb{Q}_\ell) \to H^{2r}_{\mathfrak{T}}(\mathbf{M}_\mathfrak{p}, \mathbb{Q}_\ell(r))$$
$$\mathrm{inc}_\star^* : H^{2r}_{\mathfrak{T}}(\mathbf{M}_\mathfrak{p}, \mathbb{Q}_\ell(r)) \to H^{2r}_{\mathfrak{T}}(\mathbf{B}_\mathfrak{p}, \mathbb{Q}_\ell(r)) \simeq H^0_{\Sigma}(\mathbf{S}_\mathfrak{p}, \mathbb{Q}_\ell(r))) \simeq \mathbb{Q}_\ell[\mathrm{Sh}(V^\star, K_p^\star)]$$

Theorem 2.5.1 (Theorem 4.3.10 of [2])**.** *Suppose that $N = 2r + 1$ is odd. Then the composition*

$$\mathrm{inc}_\star^* \circ \mathrm{inc}_!^\star$$

is equal to the Hecke operator

$$\mathbf{T}^\star_{N,\mathfrak{p}} = \sum_{\delta=0}^{r} \mathbf{d}_{r-\delta,p} \cdot \mathbf{T}_{N,\mathfrak{p},\delta}.$$

Here $\mathbf{T}_{N,\mathfrak{p},\delta}$ are certain 'standard Hecke operators' defined in appendix B and $\mathbf{d}_{r-\delta,p}$ are integers defined in Section 1.3 (they are basically polynomials in p).

Proof. The paper [2] cites Theorem 9.3.5 of [4], which does not exist (as of 10-2-2020). However, it seems that that it does follow from Section 7.4 of [4], which is the computation of the intersection matrix of cycle classes of the basic locus, and the combinatorics in appendix B of [2]. Unfortunately, some of the proofs in Appendix B also refer to Section 9 of [4]. □

3 Functoriality and special morphisms

The reason (I think) that we are considering such 'weird' moduli problems, is because there is a natural map (which wouldn't be there if we just considered the usual PEL type unitary Shimura varieties)

$$\mathbf{m}_\uparrow : \mathbf{M}_\mathfrak{p}(V_n) \to \mathbf{M}_\mathfrak{p}(V_{n+1})(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p) \to (A_0, \lambda_0, \eta_0^p; A \times A_0, \lambda \times \lambda_0, \eta^p \oplus 1),$$

where now V_n, V_{n+1} are standard indefinite hermitian spaces of rank $n, n+1$ respectively. We are going to show that all constructions of the previous section are compatible with this, constructing a commutative diagram:

$$\begin{array}{ccccc} \mathbf{S}_\mathfrak{p}(V_{n+1}) & \longleftarrow & \mathbf{B}_\mathfrak{p}(V_{n+1}) & \longrightarrow & \mathbf{M}_\mathfrak{p}(V_{n+1}) \\ \uparrow & & \uparrow & & \\ \mathbf{S}_\mathfrak{p}(V_n)_{\mathrm{sp}} & \longleftarrow & \mathbf{B}_\mathfrak{p}(V_n)_{\mathrm{sp}} & & \uparrow \mathbf{m}_\uparrow \\ \downarrow & & \downarrow & & \\ \mathbf{S}_\mathfrak{p}(V_n) & \longleftarrow & \mathbf{B}_\mathfrak{p}(V_n) & \longrightarrow & \mathbf{M}_\mathfrak{p}(V_n) \end{array} \tag{2}$$

Definition 3.0.1. *We define $\mathbf{S}_\mathfrak{p}(V_n)_{sp} = \mathbf{S}_\mathfrak{p}(V_n, K^p)_{sp}$ to be the functor whose S points is the set of of equivalence classes of tuples*

$$(A_0, \lambda_0, \eta_0^p; A^\star, \lambda^\star, \eta^{p\star}; A^\star_\natural, \lambda^\star_\natural, \eta^{p\star}_\natural; \delta^\star),$$

where

- $(A_0, \lambda_0, \eta_0^p; A^\star, \lambda^\star, \eta^{p\star}) \in \mathbf{S}_\mathfrak{p}(V_n)(S)$;
- $(A_0, \lambda_0, \eta_0^p; A_\natural^\star, \lambda_\natural^\star, \eta_\natural^{p\star}) \in \mathbf{S}_\mathfrak{p}(V_{n+1})(S)$;
- $\delta^\star : A^\star \times A_0 \to A_\natural^\star$ *is an* $\mathcal{O}_F$*-linear quasi* p*-isogeny such that*
    1. $\ker \delta^\star[p^\infty] \subset (A^\star \times A[\mathfrak{p}])$;
    2. $\lambda^\star \times \varpi\lambda_0 = \delta^{\star\vee} \circ \lambda_\natural^\star \circ \delta^\star$;
    3. $\delta^\star$ *is compatible with the* K^p *level structures.*

Lemma 3.0.2. *The forgetful map*

$$\mathbf{S}_\mathfrak{p}(V_n)_{sp} \to \mathbf{S}_\mathfrak{p}(V_{n+1})$$

is an isomorphism when n *is odd and finite etale of degree* $p+1$ *when* n *is even.*

Definition 3.0.3. *We now define* $\mathbf{B}_\mathfrak{p}(V_n)_{sp}$ *by the following Cartesian diagram:*

$$\begin{array}{ccc} \mathbf{B}_\mathfrak{p}(V_n)_{sp} & \longrightarrow & \mathbf{S}_\mathfrak{p}(V_n)_{sp} \\ \downarrow & & \downarrow \\ \mathbf{B}_\mathfrak{p}(V_n) & \longrightarrow & \mathbf{S}_\mathfrak{p}(V_n). \end{array}$$

One can now check that there is indeed a morphism $\mathbf{B}_\mathfrak{p}(V_n)_{\mathrm{sp}} \to \mathbf{B}_\mathfrak{p}(V_{n+1})$ making (2) commute.

References

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